The Loaded Quality factor.

If we consider a resonant circuit with a transmission line coupling circuit with characteristic impedance $Z_0$ and infinite length the quality factor is modified as follows

$$Q_L = \frac{Q_U}{1 + \beta} = \frac{Q_U}{1 + \frac{Z_0}{R}}$$  \hspace{1cm} (1)

For the RLC series circuit.

The trade of the magnitude for the admittance can be determined for a given $Q_U$ in order to derive the frequencies that indicate the 3dB bandwidth of the resonant circuit. This may be done by expressing the circuit impedance in function of the quality factor.

$$Z_{eq} = R + sL + \frac{1}{sC} \rightarrow R + j\left(\omega L - \frac{1}{\omega C}\right)$$  \hspace{1cm} (2)

$$Z_{eq} = R\left[1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega CR}\right)\right]$$

By multiplying and dividing the imaginary part for the resonant frequency we achieve

$$Z_{eq} = R\left[1 + j\left(\frac{\omega L}{R} - \frac{\omega_r}{\omega CR}\right)\right]$$

$$\rightarrow R\left[1 + j\frac{L\omega_r}{R} \left(\frac{\omega}{\omega} - \frac{\omega_r}{\omega}\right)\right]$$

$$= R\left[1 + jQ_U \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right]$$

The system poles are derived by stimulating the circuit with a voltage generator and by reading the current namely at the admittance function.

$$Y_{eq} = \left[R\left[1 + jQ_U \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right]\right]^{-1}$$

$$= \frac{G}{1 + jQ_U \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

The abs value is
\[ |Y_{eq}| = \frac{|G|}{\sqrt{1 + \left( Q_U \left( \frac{\omega}{\omega_r} - \frac{\omega}{\omega} \right) \right)^2}} \quad (3) \]

And the 3 dB bandwidth is that for which the magnitude is

\[ 3dB \rightarrow \log^{-1}(\frac{3}{20}) = \sqrt{2} \]

Namely

\[ 1 + Q_U^2 \left( \frac{\omega}{\omega_r} - \frac{\omega}{\omega} \right)^2 \]

\[ = 2 \rightarrow Q_U^2 \left( \frac{\omega}{\omega_r} - \frac{\omega}{\omega} \right)^2 \quad (4) \]

\[ = 1 \rightarrow Q_U \left( \frac{\omega}{\omega_r} - \frac{\omega}{\omega} \right) = \pm 1 \]

Developing the (4) as a polynomial second grade expression is derived that allows to extract two frequency values for which the magnitude of the transfer function is reduced of 3 dB

\[ Q_U \left( \frac{\omega^2 - \omega_r^2}{\omega \omega_r} \right) = \pm 1 \]

\[ Q_U \omega^2 - \omega \omega_r - Q_U \omega_r^2 = 0 \quad (5) \]

\[ Q_U \omega^2 + \omega \omega_r - Q_U \omega_r^2 = 0 \quad (6) \]

The solutions of equation (6):

\[ \omega_{1,2} = \omega_r \pm \sqrt{\omega_r^2 + \frac{4Q_U^2 \omega_r^2}{4Q_U^2 - 1}} \]

\[ = \omega_r \left( 1 \pm \sqrt{\frac{1 + 4Q_U^2}{4Q_U^2}} \right) \]

\[ = \omega_r \left( \frac{1}{2Q_U} \pm \sqrt{\frac{1}{4Q_U^2} + 1} \right) \]
Whereas that of (6)

\[ \omega_{1,4} = -\omega_r \pm \sqrt{\omega_r^2 + 4Q_u^2 \omega_r^2} \]

\[ = \omega_r \left( -\frac{1}{2Q_u} \pm \sqrt{\frac{1}{4Q_u^2} + 1} \right) \]

If we put in a univocal term the positive solutions we found

\[ \omega_{1,2} = \omega_r \left( \sqrt{\frac{1}{4Q_u^2} + 1} \pm \frac{1}{2Q_u} \right) \] (7)

That indicated as the frequency are not mirrored compared to the circuit frequency. The magnitude of the transfer function can be drawn in function of the frequency and it has a bell trade with the frequency that indicate the 3 dB band given by (7). The trade of this curve is more or less high in function of the \( Q_U \) value.

![Figure 1](image-url)

Since a resonant circuit can be seen as a bipole with Z impedance then it is possible to determinate the trade of the reflection coefficient on the Smith Chart. For a give characteristic impedance \( Z_0 \) we can write

\[ \Gamma = \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0} = \frac{(R-Z_0) + j(\omega L - \frac{1}{\omega C})}{(R+Z_0) + j(\omega L - \frac{1}{\omega C})} \]

\[ = \frac{\left[ (R-Z_0) + j(\omega L - \frac{1}{\omega C}) \right] \left[ (R+Z_0) - j(\omega L - \frac{1}{\omega C}) \right]}{(R+Z_0)^2 + \left( \omega L - \frac{1}{\omega C} \right)^2} \] (8)
\[ R^2 - Z_0^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2 \]

\[ \Re(\Gamma) = \frac{R^2 - Z_0^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2}{(R + Z_0)^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2} \]

\[ \Im(\Gamma) = \frac{2Z_0 \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)}{(R + Z_0)^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2} \]

The expressions for module and phase are derived as:

\[ |\Gamma| = \sqrt{\left( \frac{R^2 - Z_0^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2}{(R + Z_0)^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2} \right)^2 + \left( \frac{2Z_0 \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)}{(R + Z_0)^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2} \right)^2} \] (9)

\[ \angle \Gamma = \arctan \left( \frac{2Z_0 \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)}{R^2 - Z_0^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2} \right) \] (10)

These trade given information about how the reflection coefficient moves as the frequency value increases its value. While as the frequency value increase the reflection coefficient move in a constant circumference which equation can be derived by developing the previously equation, the imaginary part gives information about which the trajectories.

\[ \lim_{\omega \to \infty} \arctan \left( \frac{2Z_0 \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)}{R^2 - Z_0^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R + Z_0} \right)^2} \right) \]

\[ \to \arctan \left( \frac{\infty}{\infty^2} \right) \to 0^+ \]
\[
\lim_{\omega \to 0} \arctan \left( \frac{2Z_0 \left( \omega L - \frac{1}{\omega C} \right)}{R^2 - Z_0^2} \right) \\
\to \arctan \left( \frac{-\infty}{\infty^2} \right) \rightarrow 0^-
\]

While the resonance frequency the reflection coefficient assumes a purely real value therefore the phase is

\[
\angle \Gamma|_{\omega = \omega_r} = \arctan \left( \frac{0}{R^2 - Z_0^2} \right)
\]

\[
\begin{cases}
\arctan \left( \frac{0}{R^2 - Z_0^2} \right) = 0 & \text{if } R > Z_0 \\
\arctan \left( \frac{0}{R^2 - Z_0^2} \right) = \pm 180 & \text{if } R < Z_0
\end{cases}
\]

This implies that as the frequency increases a reflection coefficient for a series RLC circuit turns in clockwise sense on the Smith chart, and that cross the horizontal axes at the frequency value \(\omega_r\).

Then it is possible to define the \(Q_U\) of series resonant circuit after having introduced the yellow curve represented in the figure. In many textbooks this curve is defined as the curve drawn when the reactive and resistive parts of impedances are equals. It is represented by two arcs of circumference that intercept each others and with the unitary circumference which centres stay on imaginary axis and measure \(2^{1/2}\). By referring to the \(Q\) definition as factionary number between reactive and resistive part of an impedance value this would represent the well known \(Q\) contour curve. In
literature is useful the Q contour = 1. It is possible also to define the reciprocal curve because the Q is a unitary ratio

\[ Q_U \left( \frac{\omega}{\omega_r} - \frac{\omega}{\omega} \right) = \pm 1 \]

The two frequencies \( \omega_1 \) e \( \omega_2 \) represent the solution of previously relationship, the solutions of previously equation are the interception with +1 (higher curve Eye) and -1 (lower arc).
These frequencies indicate the frequencies of the 3dB band for a given resonant circuit, whereas the frequency for which the trade of reflection coefficient intercept the horizontal axis represent the resonant frequency of the circuit.
The definition of \( Q_U \) then can be applied and may be better used as check condition in design phase. In good textbooks i.e. [1] in fact the quality factor of resonant unloaded circuit is defined as

\[ Q_U = \frac{\omega_r}{\omega_2 - \omega_1} \quad (11) \]

As the reflection coefficient moves in clockwise sense as the frequency increases, the condition \( \omega_1 < \omega_r < \omega_2 \) it will be verified.

**REFERENCES**