An equivalence criterion
for PL-manifolds (*)

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Abstract. Aim of the paper is to translate the homeomorphism problem for n-dimensional PL-manifolds, with or without boundary, into an equivalence problem for pseudosimplicial triangulations (i.e. a suitable generalization of simplicial triangulations, where two (curvilinear) simplices may intersect in more than one face), by means of a finite number of moves, called (geometric) dipole moves. Note that the environment of the present work is closely related to the representation method for PL-manifolds via edge-coloured graphs, since \((n+1)\)-coloured graphs representing n-manifolds are a «discrete way» to visualize suitable pseudosimplicial triangulations. From the graph-theoretical point of view, the equivalence problem was already faced – and solved – in [FG] for closed n-manifolds and in [C] in the general 3-dimensional setting; here, the equivalence criterion via (geometric) dipole moves is proved to hold for the whole class of PL n-manifolds; moreover, it is proved to be equivariant with respect to boundary triangulation.


1. INTRODUCTION

The problem of deciding whether two different «objects» do represent the same manifold plays a crucial role in every topological-combinatorial representation theory of PL-manifolds\(^{(1)}\). From this viewpoint, the present paper takes into account pseudosimplicial

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\(^{(1)}\) For instance, we recall fundamental results of [R] and [S] (resp. of [K]) (resp. of [M]) (resp. of [Pi]) concerning Heegaard diagrams of 3-manifolds (resp. framed links) (resp. generalized Heegaard diagrams) (resp. simple 3-coverings of \(S^3\) branched over links).

complexes triangulating $n$-dimensional PL-manifolds, with or without boundary: roughly speaking, they may be described as a suitable generalization of simplicial triangulations, where two (curvilinear) simplices possibly intersect in more than one face.

This paper allows to translate the homeomorphism problem for the represented manifolds into an equivalence problem for (coloured) pseudosimplicial triangulations, by means of a finite set of moves, called (geometric) dipole moves.

Note that the environment of the present work is closely related to the representation method for PL-manifolds via edge-coloured graphs, since any $(n + 1)$-coloured graph representing an $n$-manifold $M$ simply «visualizes in a discrete way» a suitable pseudosimplicial triangulation of $M$.

We point out that, as far as the graph-theoretical point of view is concerned, the equivalence problem was already faced – and solved – in [FG], but for closed $n$-manifolds only; further, a different approach has been used in [C 2 ] to complete the effort for the whole class of 4-coloured graphs representing 3-manifolds, but only partial results have been achieved in the general $n$-dimensional setting.

Here, the equivalence criterion via (geometric) dipole move is proved to hold for the whole class of PL $n$-manifolds, by making use of an approach that combines both the method on which [FG] is based (i.e. cone algorithm) and the theory on shelling and bistellar operations involved in [C 2 ]. Moreover, in case of PL $n$-manifolds with non empty boundary, the criterion is proved to be equivariant with respect to boundary triangulation (2).

The author hopes the described results to open new possibilities concerning definition and testing of $n$-dimensional invariants for PL-manifolds (with or without boundary), by means of an approach similar to that used in [KL; Theorem 11] to verify the topological invariance of Turaev-Viro invariant for 3-manifolds.

2. PSEUDOCOMPLEXES, COLOURED GRAPHS AND DIPOLE MOVES

Throughout the paper, we shall work in the piecewise-linear (PL) category, for which we refer to [RS].

According to [HW; page 49], an $n$-dimensional pseudocomplex is defined to be a finite collection $K$ of closed $h$-balls ($0 \leq h \leq n$), usually called $h$-simplices, so that:

- $|K| = \bigcup \{B/B \in K\} = \biguplus \{B/B \in K\}$ (where the symbol $\biguplus$ denotes disjoint union);
- if $A, B \in K$, then $A \cap B$ is a (possibly void) union of balls of $K$;
- for each $h$-ball $B \in K$, the subset $\{B' \in K / B' \subset B\}$, ordered by inclusion, is isomorphic with the lattice of all faces of the standard $h$-simplex.

If $M$ is a (PL) $n$-manifold, with or without boundary, then a pseudosimplicial

(2) Note that the question about the existence of an equivariant version of the equivalence criterion via dipole moves was already raised – as an open problem for 3-manifolds – in [C 2 , page 133].
triangulation of $M^\sigma$ is any $n$-dimensional pseudocomplex $K$ such that $|K| = M^\sigma$.

In order to characterize $n$-dimensional pseudocomplexes representing manifolds, the following definitions are of use:

**Definition 1.** If $\sigma$ is a simplex of an $n$-dimensional pseudocomplex $K$, the disjoint star of $\sigma$ in $K$, $\text{std}(\sigma; K)$, is the subcomplex of $K$ consisting of the disjoint union of the $n$-simplices containing $\sigma$ and of their proper faces, with re-identification of the faces containing $\sigma$ and of their faces.

**Definition 2.** If $\sigma$ is a simplex of an $n$-dimensional pseudocomplex $K$, the disjoint link of $\sigma$ in $K$, $\text{lkd}(\sigma; K)$, is the subcomplex of $\text{std}(\sigma; K)$ consisting of simplices disjoint from $\sigma$.

It is not difficult to prove that an $n$-pseudocomplex $K$ represents a PL $n$-manifold $M^n = |K|$ (or, in other words, $K$ is a pseudosimplicial triangulation of $M^n$) if and only if, for any vertex $v \in K$, $\text{lkd}(v; K)$ is either a $(n-1)$-sphere (in this case, $v$ is said to be an internal vertex) or a $(n-1)$-ball (in this case, $v$ is said to be a boundary vertex).

**Definition 3.** A $n$-pseudocomplex $K$ is said to be colourable if there exists a map $\xi : S_0(K) \to \Delta_n = \{0, 1, \ldots, n\}$ ($S_0(K)$ being the vertex set of $K$), which is injective on every simplex. The pair $(K, \xi)$ is said to be a coloured $n$-pseudocomplex.

From now on, let $\mathcal{K}_n$ denote the class of all pseudosimplicial triangulations of $n$-manifolds, which admit a vertex labelling $\xi : S_0(K) \to \Delta_n = \{0, 1, \ldots, n\}$ so that $\xi(v) \in \Delta_{n-1}$ for every $v \in \partial K$.

Note that the class $\mathcal{K}_n$ results to be a universal representing tool for PL $n$-manifolds, since, for every $n$-manifold $M^n$, the existence of a pseudocomplex $K \in \mathcal{K}_n$ representing $M^n$ may be directly proved: it is sufficient to consider the first baricentric subdivision of any simplicial triangulation of $M^n$, and label every vertex by the dimension of the simplex whose barycenter it is.

Now, a coloured triangulation $K \in \mathcal{K}_n$ may be combinatorially visualized by means of an $(n+1)$-coloured graph $(\Gamma, \gamma)$, whose underlying multigraph $\Gamma = \Gamma(K)$ is nothing but the dual graph of $K$, i.e. the 1-skeleton of the ball-complex dual to $K$, and the edge-colouring $\gamma : E(\Gamma) \to \Delta_n$ is induced by vertex-labelling of $K$: $(\Gamma, \gamma)$ has a vertex $\gamma(\sigma)$ for each (labelled) $n$-simplex $\sigma \in K$, and an $i$-coloured edge $(i \in \Delta_n)$ connecting $\gamma(\sigma)$ and $\gamma(\tau)$ for every pair $\sigma, \tau$ of $n$-simplices of $K$ sharing the $(n-1)$-face opposite to $i$-labelled vertex.

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(3) The manifold representation theory via edge-coloured graphs – known also as crystallization theory – was firstly introduced by M. Pezzana and his Italian school (see [FGG] and [BCG], together with their references); further, it has been developed by many other researchers, too (see [BM], [LiM], [V], [CV], [Li], [KL]). We shall repeat here few basic notions useful for this paper, in order to make it essentially self-contained. As far as elementary terminology of graph theory is concerned, we refer to [W].
Note that the properties of the class $\mathcal{K}_n$ directly imply the coloured graph $\Gamma = \Gamma(K)$ to be \textit{regular with respect to the «last» colour} $n$: this means that the degree of any vertex $v \in V(\Gamma)$ is either $n+1$ (in case $v = v(\sigma)$, with $\sigma \in K$ having no boundary $(n-1)$-face, so that $v$ is adjacent to exactly one $i$-coloured edge, $\forall i \in \Delta_n$) or $n$ (in case $v = v(\sigma)$, with $\sigma \in K$ having a boundary $(n-1)$-face, so that $v$ is adjacent to exactly one $i$-coloured edge, $\forall i \in \Delta_n$, but to no $n$-coloured edge\(^{18}\)).

Obviously, not every $(\Gamma, \gamma)$ satisfying the above properties\(^{18}\) represents a pseudocomplex $K = K(\Gamma) \in \mathcal{K}_n$ triangulating an $n$-manifold $M$: since the disjoint link of any $c$-labelled vertex of $K$ (resp. of any $m$-simplex of $K$ whose vertices are labelled by $\{c_1, c_2, \ldots, c_{m+1}\}$) is represented by a $\hat{c}$-residue of $\Gamma$, i.e. a connected component of $\Gamma' = (V(\Gamma), \gamma'(\Delta_n - \{c\}))$ (resp. by a $(\Delta_n - \{c_1, c_2, \ldots, c_{m+1}\})$-residue of $\Gamma$, i.e. a connected component of $(V(\Gamma), \gamma'(\Delta_n - \{c_1, c_2, \ldots, c_{m+1}\}))$), the necessary and sufficient condition is that each $\hat{c}$-residue, $\forall c \in \Delta_n$, represents either a $(n-1)$-ball or a $(n-1)$-sphere.

It is now possible to define, both in the pseudosimplicial setting and in the graph-theoretical one, the set of graph-moves we are interested in.

\textbf{Definition 4.} Let $K$ be a pseudosimplicial triangulation of an $n$-manifold $M'$, consisting of at least two $n$-simplices. A \textit{geometric} $h$-dipole $(1 \leq h \leq n)$ of $K$ is an $n$-dimensional pseudocomplex $\mathcal{D}$ (subcomplex of $K$) satisfying the following properties:

i) $\mathcal{D}$ consists of two $n$-simplices $\sigma, \sigma'$ having $h$ common $(n-1)$-faces $F_1, F_2, \ldots, F_h$;

ii) if $A^{h-1} = \sigma^{h-1} \cap F_i$ and $B^{h-1}$ (resp. $C^{h-1}$) is the $(h-1)$-face of $\sigma$ (resp. $\sigma'$) opposite to $A^{h-1}$, then $B^{h-1}$ is different from $C^{h-1}$, and no simplex $\tau \in K$ exists, having both $B^{h-1}$ and $C^{h-1}$ as faces;

iii) if either $\sigma$ or $\sigma'$ has no boundary $(n-1)$-face, then either $B^{h-1}$ or $C^{h-1}$ is an internal simplex (i.e. either $\operatorname{lkd}(B^{h-1}; K)$ or $\operatorname{lkd}(C^{h-1}; K)$ triangulates an $(n-h)$-sphere).

In particular, a geometrical dipole $\mathcal{D} = \{ \sigma, \sigma' \}$ will be said to be an internal dipole if either $\sigma$ or $\sigma'$ has no boundary $(n-1)$-face.

\textbf{Definition 5.} The \textit{elimination} (or \textit{cancellation}) of the (geometrical) $h$-dipole $\mathcal{D}$ in $K$ consists of:

a) deleting $\mathcal{D}$ from $K$;

b) identifying the $(n-1)$-face $\tau$ of $\sigma$ with the $(n-1)$-face $\tau'$ of $\sigma'$, whenever $\tau, \tau'$, are opposite to the same vertex $v \in A^{h-1}$ and none of them is a boundary face in $K$.

The \textit{insertion} of a (geometrical) $h$-dipole is the inverse process; by a (\textit{geometric}) dipole move (resp. \textit{internal} (geometric) dipole move) we mean either the elimination or the

\(^{18}\) In this case, $v \in V(\Gamma)$ is said to be an \textit{internal} vertex.

\(^{18}\) In this case, $v \in V(\Gamma)$ is said to be a \textit{boundary} vertex.

\(^{18}\) According to standard notation, we denote by $\mathcal{D}_n$ the class of edge-coloured graphs regular with respect to colour $n$. 
insertion of a (geometric) $h$-dipole (resp. internal (geometric) $h$-dipole), for some $1 \leq h \leq n$.

**Definition 6.** Let $(\Gamma, \gamma) \in \mathcal{C}_n$ be an $(n+1)$-coloured graph with $\# V(\Gamma) > 2$. An $h$-dipole $(1 \leq h \leq n)$ of $\Gamma$ is a subgraph $\Theta$ consisting of two vertices $v, w \in V(\Gamma)$ joined by $h$ edges coloured by $c_1, c_2, \ldots, c_h \in \Delta_n$, and satisfying the following conditions:

a) $v$ and $w$ belong to different $(\Delta_n - \{c_1, c_2, \ldots, c_h\})$-residues of $(\Gamma, \gamma)$, $\Xi_v, \Xi_w$ say;

b) if either $v$ or $w$ is an internal vertex, then either $\Xi_v$ or $\Xi_w$ is a regular $(n+1-h)$-coloured graph.

The colours $c_1, c_2, \ldots, c_n$ are said to be involved in the dipole $\Theta = \{v, w\}$. Moreover, $\Theta = \{v, w\}$ is said to be an internal dipole if either $v$ or $w$ is an internal vertex.

Note that, if $(\Gamma, \gamma)$ is a regular $(n+1)$-coloured graph and/or colour $n$ is involved in the dipole $\Theta$ and/or $h = n$, then condition b) is always satisfied.

**Definition 7.** The elimination (or cancellation) of the $h$-dipole $\Theta$ in $(\Gamma, \gamma)$ consists of:

a) deleting $\Theta$ from $(\Gamma, \gamma)$;

b) welding the «hanging» pairs of edges of the same colour $c \in \Delta_n - \{c_1, c_2, \ldots, c_h\}$.

The insertion of an $h$-dipole is the inverse process; by a dipole move (resp. internal dipole move) we mean either the elimination or the insertion of an $h$-dipole (resp. internal $h$-dipole), for some $1 \leq h \leq n$.

Note that $\Theta$ is an $h$-dipole (resp. internal $h$-dipole) in an $(n+1)$-coloured graph $(\Gamma, \gamma) \in \mathcal{C}_n$ representing a manifold if and only if $D = K(\Theta)$ is a (geometric) $h$-dipole (resp. internal (geometric) $h$-dipole) in $K = K(\Gamma) \in \mathcal{K}_n$.

As pointed out in [C3, Prop. 1], dipole move on a pseudosimplicial triangulation of an $n$-manifold yields a new pseudocomplex triangulating the same $n$-manifold (in fact, dipole insertion is a (trivial) connected sum between $M^n$ and the $n$-ball $|D|$); in particular, internal dipole moves do not affect the boundary triangulations, too. Hence, we can state:

**Proposition 1 [C3].** If $K$ is a pseudocomplex triangulating an $n$-manifold $M^n$, and $K'$ is obtained from $K$ by a dipole move, then $K'$ triangulates $M^n$ too; further, if the dipole move is an internal one, then $\partial K = \partial K'$.

Our main result (see the Main Theorem, at the beginning of the fifth paragraph) states that, if the pseudocomplexes triangulating manifolds are associated to $(n+1)$-coloured graphs, the converse is true, too: $K = K(\Gamma)$ and $K' = K(\Gamma')$, both triangulating the $n$-manifold $M^n$, are always equal up to dipoles (i.e. a finite number of dipole moves exists, yielding $K'$ (resp. $K$) from $K$ (resp. $K'$)). Further, if $\partial K = \partial K'$, then $K$ and $K'$ are equal up to internal dipoles (i.e. a finite number of internal dipole moves exists, yielding $K'$ (resp. $K$) from $K$ (resp. $K'$)): this proves our equivalence criterion for PL-manifolds to be equivariant with respect to the boundary triangulation.
3. MOVES ON SIMPLICIAL TRIANGULATIONS OF MANIFOLDS

In the present paragraph, \( K \) will denote a simplicial triangulation of a PL \( n \)-manifold \( M^n \) (with possibly void boundary, if not otherwise stated). As usual, if \( A \in K \) is an arbitrary simplex, we set:

\[
\text{st}(A; K) := \{ C \in K \mid \exists B \in K, C \subseteq B, B \supseteq A \}; \\
\text{lk}(A; K) := \{ C \in K \mid C \in \text{st}(A; K), C \cap A = \emptyset \}.
\]

The notion of bistellar operation was originally defined in 1986 by U. Pachner (see [P]):

**Definition 8.** [P] Let \( A \neq \emptyset \) be a \( k \)-simplex (\( 0 \leq k \leq n \)) of \( K \), such that \( \text{lk}(A; K) \) is the boundary complex \( \partial B \) of an \( (n-k) \)-simplex \( B \) not contained in \( K \). Then, **bistellar \( k \)-operation** \( \chi(A,B) \) on \( K \) is the process yielding

\[
\chi(A,B) := (K - A \ast \partial B) \cup \partial A \ast B
\]

where \( \ast \) denotes the join of two simplicial complexes.

Note that \( \chi^{-1}(A,B) = \chi(B,A) \) is a bistellar \((n-k)\)-operation; moreover, it is easy to check that bistellar operations do not affect the homeomorphism class of the triangulated manifold. Hence, we can say that two simplicial triangulations \( K, K' \) of the same manifold are **bistellar equivalent** if they can be obtained from each other by a finite sequence of bistellar operations.

The importance of bistellar operations relies on their capability of solving the equivalence problem for PL \( n \)-manifolds, both in the closed case (see [P]) and in the case of simplicial triangulations conciding on their non-void boundary (see [C.1]):

**Proposition 2.** [P] [C.1.] Let \( K, K' \) be simplicial triangulations of \( n \)-manifolds, with \( \partial K = \partial K' \) (possibly void). Then, \( |K|, |K'| \) are PL-homeomorphic if and only if \( K, K' \) are bistellar equivalent.

We conclude the paragraph with a glance toward «shellability theory», a survey of which is contained in [DK.2]; here, we only recall definitions and results strictly necessary for our proofs.

**Definition 9.** A (pure finite) simplicial complex \( K \) is said to be **shellable** if an ordering \((\sigma_1, \sigma_2, \ldots, \sigma_N)\) of its \( n \)-simplices exists, such that for every \( j = 2, \ldots, N \) the intersection \( \sigma_j \cap (\bigcup_{i=1}^{j-1} \sigma_i) \) is a non-empty union of \((n-1)\)-faces of \( \sigma_j \).

Since \( \sigma_j \cap (\bigcup_{i=1}^{j-1} \sigma_i) \) turns out to be an \((n-1)\)-dimensional ball or sphere, it easily follows that if \( K \) is a shellable \( n \)-complex triangulating a manifold \( M^n \), then \( M^n \) is either \( S^n \) or \( D^n \). In dimension two, also the converse is true: every simplicial 2-complex...
triangulating $\mathbb{S}^2$ or $\mathbb{D}^2$ is shellable (see [DK,1]). On the contrary, in general dimension, the existence of non-shellable PL balls and spheres has been proved (see [Ru] and [L], for example). Notwithstanding this, the notion of bistellar operation leads to the following important result:

**Proposition 3** [P]. For every simplicial triangulation $K$ of $\mathbb{S}^n$, there exists a shellable simplicial triangulation $\bar{K}$ of $\mathbb{D}^{n+1}$, such that $\partial \bar{K} = K$.

### 4. Cone Algorithms

The notion of cone-algorithm has been introduced for closed $n$-manifolds in [FG] and – among other – gave rise to an alternative proof of Pezzana fundamental theorem about universality of crystallization theory (see [FGG] or [BCG]); the present paragraph is devoted to extend to the general situation only a particular case of cone-algorithm (actually, the case $i = 0$, in the notations of [FG]), which will be useful for our purposes.

**Definition 10.** A cone-vertex of an $n$-dimensional pseudocomplex $K$ is a vertex $v$ which belongs to each $n$-simplex of $K$ (i.e. $st(v; K) = K$ holds).

Let $M^n$ be a (connected) PL $n$-manifold; $\mathcal{C}(M^n)$ will denote the class of all pseudocomplexes triangulating $M^n$, while $\mathcal{C}_1(M^n)$ will denote the subclass of $\mathcal{C}(M^n)$ whose elements contain (at least) an inner cone-vertex.

If $K$ belongs to $\mathcal{C}_0(M^n)$, then it may give rise to a new pseudocomplex $K' \in \mathcal{C}_1(M^n)$ by means of the following process (cone-algorithm):

(a) Arbitrarily choose a spanning tree $T$ of the dual graph $\Lambda(K)$.

(b) Attach together the $n$-simplices of $K$ (corresponding to the vertices of $T$) by identification of the $(n-1)$-faces which are dual to the edges of $T$, so that a pseudocomplex $D$-triangulating the $n$-ball $\mathbb{D}^n$ is constructed. Note that every vertex of $D$ lies on the boundary $\Sigma := \partial D$, and that the starting pseudocomplex $K$ could be obtained from $D$ by identification of suitable pairs of $(n-1)$-simplices of $\Sigma$ (called twin $(n-1)$-simplices).

(c) Let $D'$ be the $n$-pseudocomplex (triangulating the $n$-ball $\mathbb{D}^n = |D|$) carried out from $D$ by making the join from an arbitrarily chosen inner point $w$ over its (unaffected) boundary $\Sigma = \partial D$.

(d) The required pseudocomplex $K' \in \mathcal{C}_1(M^n)$ – having $w$ as inner cone-vertex – is simply obtained from $D' = w \ast \Sigma$ by identifying twin $(n-1)$-simplices of $\Sigma$. In other words, there is a canonical projection $p : w \ast \Sigma \to K'$, defined by $p(\alpha) = p(\beta)$ for every pair $(\alpha, \beta)$ of twin $(n-1)$-simplices of $\Sigma$ and $p(\sigma) = \sigma$ for every $(n-1)$-simplex $\sigma \in \partial K$.

From now on, we will denote by $\mathcal{U}(K)$ the subset of $\mathcal{C}_1(M^n)$ consisting of all pseudocomplexes $K'$ obtained from $K \in \mathcal{C}_0(M^n)$ by a cone-algorithm; in particular, since the choice of the spanning tree $T$ in $\Lambda(K)$ uniquely determines the process, we will say that $K' \in \mathcal{U}(K)$ is obtained from $K$ by the cone-algorithm based on $T$, and we will use the
notation $K' = A_p(K)$. Note that, by construction itself, for every $K' \in \mathcal{U}(K)$, $\partial K' = \partial K$.

The relationship between the elements of $\mathcal{U}(K)$ is described in the following:

**Lemma 4.** Let $K \in \mathcal{C}(\mathcal{M})$. If $K', K'' \in \mathcal{U}(K)$, then $K', K''$ are equal up to internal dipoles.

**Proof.** First, let us assume $K' = A_p(K)$ and $K'' = A_p(K)$, where $T'$ (resp. $T''$) is obtained from the (disjoint) union of two subtrees $T_i, T_j$ of $\Lambda(\Gamma)$ by adding the edge $e'$ (resp. $e''$). Further, let $\Sigma_i$ (resp. $\Sigma_j$) be the boundary of the $n$-ball $D_i$ (resp. $D_j$) associated to $T_i$ (resp. $T_j$) by step (b) of cone-algorithm. If $\pi$ is the identification map on $\Sigma_i \cup \Sigma_j$ yielding $K$ from $D_i \cup D_j$, then obviously $K = (w_i * \Sigma_i) \cup (w_j * \Sigma_j)$ is a new pseudo-triangulation of $\mathcal{M}$ ($w_i$ and $w_j$ being arbitrarily chosen inner points of $D_i$ and $D_j$ respectively). Now, it is easy to check that, if $\sigma'$ (resp. $\sigma''$) is the $(n-1)$-simplex dual to the edge $e'$ (resp. $e''$), then $w_i * \sigma'$ (resp. $w_j * \sigma''$) and $w_j * \sigma'$ (resp. $w_i * \sigma''$) constitute an internal 1-dipole $D_i$ (resp. $D_j$) in $K$. Thus, since $K' = A_p(K)$ (resp. $K'' = A_p(K)$) is obtained from $K$ by eliminating $D_i$ (resp. $D_j$), the statement is proved in the particular case of the assumption; the general proof easily follows by induction on the number of edges contained in $T'$ but not in $T''$. 

As pointed out in [FG] and [F] for the closed case, it is not difficult to see that – under certain assumptions, related to the identification map $p – a$ dipole in the $(n-1)$-pseudo-complex $\Sigma$ may give rise to a dipole in the resulting $n$-pseudo-complex $K = p(w * \Sigma)$. In particular, we have:

**Lemma 5.** Let $\mathcal{D} = \{\sigma, \tau\}$ be an $h$-dipole of $\Sigma$, and let $\Sigma'$ be the $(n-1)$-pseudo-complex obtained from $\Sigma$ by cancelling $\mathcal{D}$; further, let $F_1, \ldots, F_h$ be the common $(n-2)$-faces of $\sigma$ and $\tau$, and let $B^{n-1}$ (resp. $C^{n-1}$) be the $(h-1)$-face of $\sigma$ (resp. $\tau$) opposite to $A^{n-1} = \cap_{i=1}^h F_i$.

(a) If $\sigma$ and $\tau$ are twin (i.e., $p(\sigma) = p(\tau)$), then $p(w * \mathcal{D})$ is an internal $(h+1)$-dipole in $K = p(w * \Sigma)$, whose elimination gives rise exactly to the $n$-pseudo-complex $K' = p(w * \Sigma')$.

(b) If $p(\sigma) = $and $p(\tau) = $, with $p(B^{n-1}) \neq p(C^{n-1})$, then $p(w * \mathcal{D})$ is a boundary $h$-dipole in $K = p(w * \Sigma)$, whose elimination gives rise exactly to the $n$-pseudo-complex $K' = p(w * \Sigma')$.

**Proof.** Case (a): By construction, $p(w * \mathcal{D})$ consists of the $n$-simplices $w * \sigma, w * \tau$, having $w * F_1, \ldots, w * F_h, p(\sigma) = p(\tau)$ as common $(n-1)$-faces; since their intersection is exactly $A^{n-1}$, the opposite faces result to be $w * B^{n-1}$ and $w * C^{n-1}$, which are obviously different and internal in $K$. Moreover, no $n$-simplex of $K = p(w * \Sigma)$ exists, having both $w * B^{n-1}$ and $w * C^{n-1}$ as faces, since no $(n-1)$-simplex of $\Sigma$ could contain both $B^{n-1}$ and $C^{n-1}$. The first part of the statement easily follows.

Case (b): By construction, $p(w * \mathcal{D})$ consists of the $n$-simplices $w * \sigma, w * \tau$, having
w * F_1, ..., w * F_h as common (n – 1)-faces and having respectively p(σ) and p(τ) as boundary faces; since the intersection of the common faces is exactly w * A^{n–1–h}, the opposite faces result to be p(B^{n–1}) and p(C^{n–1}), which are different in K by assumption. Moreover, no n-simplex of K = p(w * Σ) exists, having both p(B^{n–1}) and p(C^{n–1}) as faces, since no (n – 1)-simplex of Σ could contain both B^{n–1} and C^{n–1}. The statement is now completely proved.

5. MAIN RESULTS

The present paragraph is entirely devoted to prove the equivalence criterion for (colorable) pseudosimplicial triangulations of PL-manifolds, together with its equivariance properties with respect to the boundary.

**Main Theorem.** Let K ∈ K_n (resp. K’ ∈ K_n) be a (colorable) pseudosimplicial triangulation of the n-manifold M_n (resp. M'_n). Then, M_n and M'_n are PL-homeomorphic manifolds if and only if K and K’ are equal up to (geometric) dipoles. Moreover, if ∂K = ∂K’ holds, then M_n and M'_n are PL-homeomorphic manifolds if and only if K and K’ are equal up to (geometric) internal dipoles.

The first preliminary result we need, is – in author’s opinion – of its own interest; in particular it ensures that, even if the (closed) manifold triangulation associated to a regular (n + 1)-coloured graph is only a pseudosimplicial one, dipole insertions are able to make it simplicial.

**Proposition 6.** Let K be a pseudosimplicial triangulation of a compact n-manifold M_n.

(a) If M_n is closed, then a finite sequence of internal dipole insertions exists, giving rise to a simplicial triangulation K* of M_n;

(b) if M_n has non-void boundary, then a finite sequence of internal dipole insertions exists, giving rise to a pseudosimplicial triangulation K* of M_n, so that every internal simplex of K* meets any other simplex of K* in a single face (if any).

**Proof.** First, let us consider the standard (colorable) pseudosimplicial triangulation K̄ of S^n consisting of two n-simplices with identified boundaries; further, let K̄ be the baricentric subdivision of K̄. If every vertex v of K̄ is labelled by the dimension of the simplex of K whose baricenter is v, then the (coloured) pseudocomplex K̄ may be obtained by considering two copies K̄^{(1)}, K̄^{(2)} of the (coloured) baricentric subdivision of an n-simplex, and by identifying corresponding (n – 1)-faces of K̄^{(1)} and K̄^{(2)} opposite to n-labelled vertices. Moreover, it is easy to check the existence of (at least) an ordering \{ ̃σ_1, ̃σ_2, ..., ̃σ_{n+1} \} (resp. ̃σ'_1, ̃σ'_2, ..., ̃σ'_{n+1}) of n-simplices of K̄^{(1)} (resp. K̄^{(2)}), such
that $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ are corresponding $n$-simplices, $\forall i \in \{1, 2, \ldots, (n+1)!\}$, and the sequence of pairs $(\tilde{\sigma}_i, \tilde{\sigma}'_i)_{i=1,\ldots,(n+1)!-1}$ constitute subsequent internal geometric dipoles in $\tilde{K}$ (all involving colour $n$), whose elimination gives rise to the standard (coloured) triangulation $\tilde{K}$ of $\mathbb{S}^r$ (having $\{\tilde{\sigma}_i, \tilde{\sigma}'_i\}$ as $n$-simplices set).

Let now $K$ be a pseudosimplicial triangulation of the $n$-manifold $M^*$, and $\sigma_i \in K$ an internal $n$-simplex; further, let $\sigma_j \in K$ be another $n$-simplex, so that $\sigma_i, \sigma_j$ share more than one common face. If $\alpha$ is an (arbitrarily) chosen common face, in order to perform simpliciality every other common face have to be separated; for, if $\beta \neq \alpha$ is a common face with maximal dimension $\text{dim}(\beta) = n - h$, let $\tau_i, \tau_j, \ldots, \tau_r$ be $h (n - 1)$-simplices having $\beta$ (and not $\alpha$) as their face, so that cutting $K$ along $\tau_i, \ldots, \tau_r$ and along $\beta$ yields two distinct copies of $\beta$, one contained in $\sigma_i$ and the other contained in $\sigma_j$. It is not difficult to check that, in case $\beta$ being an internal face (resp. a boundary face), then cutting $K$ along $\tau_i, \ldots, \tau_r$ and along $\beta$ gives rise to a pseudosimplicial triangulation $\tilde{K}$ of $M^*$ with a spherical hole $H$ added (resp. to a pseudosimplicial triangulation $\tilde{K}$ of $M^*$, too, such that $\partial \tilde{K}$ is obtained from $\partial K$ by adding an $(n - 1)$-ball $D$).

On the other hand, let $\tilde{K}^{(i)}$ be the simplicial triangulation of $\mathbb{D}^r$ obtained by cutting $\tilde{K}$ along the $(n - 1)$-face opposite to $i$-labelled vertex of the $n$-simplex $\tilde{\sigma}_i$ of $\tilde{K}^{(i)}$, for every $i \geq n + 1 - h$: obviously, $\partial \tilde{K}^{(i)}$ is isomorphic with $\partial H \subset \partial K$ (resp. with $D \subset \partial \tilde{K}$), and a (trivial) connected sum $K'$ may be performed by identifying the face opposite to $i$-labelled vertex of the $n$-simplex $\tilde{\sigma}_i \in \tilde{K}^{(i)}$ (resp. of the $n$-simplex $\tilde{\sigma}_i \in \tilde{K}^{(i)}$, where $\tilde{\sigma}_i$ is the $n$-simplex $i$-adjacent to $\tilde{\sigma}_i$ in $\tilde{K}$) with the copy of the $(n - 1)$-simplex $\tau_i$ belonging to $\sigma_i$ (resp. with the other copy of $\tau_i$ in $\partial \tilde{K}$), for every $i \geq n + 1 - h$.

It is not difficult to check – by means of the particular geometrical properties of $\tilde{K}$ – that the sequence of pairs $(\tilde{\sigma}_i, \tilde{\sigma}'_i)_{i=1,\ldots,(n+1)!-1}$ constitute subsequent (geometrical) internal dipoles in $K'$, too. Moreover, their elimination gives rise to an internal $(n - h + 1)$-dipole, consisting of the $n$-simplices $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$, whose elimination yields exactly $K$. Hence, $K'$ is obtained from $K$ by internal dipole insertions, only; by finite iteration, this proves the statement.

The next four technical lemmas link together the notions exposed in the third and fourth paragraphs. In addition to the already introduced notational conventions, we need the following definition about shellable complexes.

Let $K$ be a shellable (possibly pseudosimplicial) complex, and let $(\sigma_1, \sigma_2, \ldots, \sigma_N)$ be an ordering of its $n$-simplices, such that $\sigma_j \cap (\bigcup_{i=1}^{j-1} \sigma_i)$ is a non-empty union of $(n - 1)$-faces $\tau'_1, \tau'_2, \ldots, \tau'_{r_j} (r_j \geq 1)$ of $\sigma_j$, for every $j = 2, \ldots, N$. Thus, a spanning tree $\bar{T}$ of the
dual graph $\Lambda(K)$ may be constructed, by choosing only the edges of $\Lambda(K)$ which are dual to the faces $\tau_i^j$, for every $j = 2, \ldots, N$.

**Definition 11.** With the above notations, $\overline{T}$ is said to be a *shelling tree* in $\Lambda(K)$.

**Lemma 7.** If $K$ is a shellable (possibly pseudosimplicial) triangulation of $D^r$ and $\overline{T}$ is a shelling tree in $\Lambda(K)$, then $\Lambda_\partial(K)$ is equal up to internal dipoles to the cone over the boundary of $K$.

**Proof.** Let $D$ be the $n$-ball associated to $\overline{T}$ by step (b) of cone-algorithm and let $\Sigma := \partial D$. For every $j = 2, 3, \ldots, N$ and for every $s = 2, 3, \ldots, r_j$ (with $r_j \leq n$, $\forall j$), $\Sigma$ contains both the $(n-1)$-face $\tau^j_i$ of $\sigma$ and its twin $(n-1)$-simplex $\tau^j_\ast = \cup_{i=1}^{n-1} \tau^j_i$. It is now easy to check that the sequence of pairs $((\tau^j_i, \tau^j_\ast))_{i=2,\ldots,r_j}$ constitute subsequent $(s-1)$-dipoles in $\Sigma$; in fact, for every $k = 1, \ldots, s-1$, $\tau^j_i$ and $\tau^j_\ast$ have $\tau^j_i \cap \tau^j_\ast$ as common $(n-2)$-face, while the (trivially internal in $\Sigma$) $(s-2)$-face of $\sigma$ opposite to $\cap_{i=1}^{n-1} \tau^j_i$ surely does not belong to $\cup_{i=1}^{n-1} \sigma_i$. The thesis now directly follows from Lemma 5 (a), since $\partial K$ is simply obtained from $\Sigma$ by cancelling as many internal dipoles (consisting of twin $(n-1)$-simplices) as the edges of $\Lambda(K)$- $\overline{T}$.

**Lemma 8.** If $K \in \mathcal{C}(M^r)$, then an $L \in \mathcal{C}(M^r)$ and a cone algorithm on it exists, giving rise to $K' \in \mathcal{U}(L)$ which is equal to $K$ up to internal dipoles.

**Proof.** Let $w$ be the inner cone-vertex of $K$, and let $\Sigma = lkd(w, K), D = std(w; K) = w \ast \Sigma$. Obviously, $\Sigma$ (resp. $D$) is a pseudosimplicial triangulation of $S^{r-1}$ (resp. $D^r$), and a canonical projection $p : D \to K$ exists, which identifies suitable pairs of $(n-1)$-simplices of $\Sigma$ (called $p$-twin simplices).

Now, Proposition 6 (a) ensures the existence of a finite sequence of internal dipole insertions in $\Sigma$, giving rise to a simplicial triangulation $\Sigma'$ of $S^{r-1}$. By identifying every pair of $(n-1)$-simplices constituting an inserted dipole, the projection $p$ may be extended to a projection $p'$ on $w \ast \Sigma'$; Lemma 5 (a) ensures that $K' := p'(w \ast \Sigma')$ is obtained from $K$ by internal dipole insertions, too.

On the other hand, since $\Sigma'$ is simplicial, a simplicial shellable triangulation $D'$ of $D^r$ exists, such that $\partial D' = \Sigma'$ (recall Proposition 3). Let $L$ be the pseudosimplicial triangulation of $M^r$ obtained from $D'$ by identification of $p'$-twin $(n-1)$-simplices of $\Sigma'$. Note that $\partial L = \partial K' = \partial K$.

It is not difficult to check that, if $\overline{T}$ is a shelling tree of $D'$, then $K' := A_\partial(L)$ is equal to $K' = p'(w \ast \Sigma')$ up to internal dipoles: in fact, $A_\partial(D')$ and $w \ast \Sigma'$ are equal up to internal dipoles by Lemma 7, while $A_\partial(L)$ (resp. $K'$) is simply obtained from $A_\partial(D')$ (resp. $w \ast \Sigma'$) by identifying $p'$-twin $(n-1)$-simplices of $\Sigma'$.
Since $K' \text{ and } K$ were proved to be equal up to internal dipoles, the statement directly follows.

**Lemma 9.** Let $K, L \in \mathcal{O}(M^*)$. If $L$ is obtained from $K$ by a bistellar operation, then there exist $K' \in \mathcal{U}(K), L' \in \mathcal{U}(L)$ which are equal up to internal dipoles.

*Proof.* Let us assume $L = \chi_{\mathcal{A}(K)}$, $A$ being a suitable $k$-simplex ($0 \leq k \leq n$) of $K$; by definition, $st(A; K)$ (resp. $st(B; L)$) is a standard triangulation of $\mathbb{D}^n$ (which we call $k$-structure (resp. $(n-k)$-structure)) consisting of $n + 2$ vertices and $n + 1 - k$ ($k + 1$) $n$-simplices. It is not difficult to check that, for every $k = 0, 1, \ldots, n$, the $k$-structure is shellable (actually, extendably shellable: see [DK 2]); thus, a shelling tree $T$ (resp. $T'$) of the dual graph $\Lambda(st(A; K))$ (resp. $\Lambda(st(B; L))$) exists.

Let now $S$ be a maximal tree in $\Lambda(K)$, which extends $\overline{T}$; obviously, $S' := (S - \overline{T}) \cup \overline{T}'$ is a maximal tree in $\Lambda(L)$. If we assume $K' := \mathcal{A}(K)$ and $L' := \mathcal{A}(L)$, the thesis easily follows from Lemma 7 and from the fact that $st(A; K)$ and $st(B; L)$ have isomorphic boundaries.

**Lemma 10.** Let $K, L \in \mathcal{O}(M^*)$. If $L$ is obtained from $K$ by a dipole move (resp. internal dipole move), then there exist $K' \in \mathcal{U}(K), L' \in \mathcal{U}(L)$ which are equal up to dipoles (resp. up to internal dipoles).

*Proof.* Let us assume $L$ being obtained from $K$ by elimination of the (geometrical) $h$-dipole $\mathcal{D} = \{\sigma, \sigma'\}$, with $h \geq 1$; thus, the $(n-1)$-faces of $\sigma$ (resp. $\sigma'$) can be denoted by $F_i, F'_i, \ldots, F_{i+h}, F_{i+h+1}, \ldots, F'_{i+h}, \ldots, F'_{n+1}$, with $F_i$ and $F'_i$ opposite to the same vertex of $A^{n-k}$, for every $i = h + 1, \ldots, n + 1$. We may assume that, if $\sigma$ (resp. $\sigma'$) has $\overline{k}_i = k_i + 1 \geq 1$ (resp. $\overline{k}_i = k_i + 1 \geq 1$) boundary faces, then they are exactly $F_{n+2-k_1}, \ldots, F'_{n+2-k_1}, \ldots, F'_{n+1}$, and, in case $\overline{k}_i \geq 1$, $F'_{n+2-k}, \ldots, F'_{n+1}$.

Let $G_i$ denote the $(n-1)$-simplex of $L$ resulting from the identification of $F_i$ and $F'_i$, for every $i = h + 1, \ldots, n + 1 - k$; it is easy to check that $G_{n+1}, \ldots, G_{n+1-\overline{k}_1-k_2}$ are internal $(n-1)$-simplices of $L$, while, in case $k_1 + k_2 \geq 1$, $G_{n+1-\overline{k}_1-k_2}, \ldots, G_{n+1}$ are boundary $(n-1)$-simplices of $L$.

Let now $T'$ be a maximal tree in the dual graph $\Lambda(L)$, which contains the edge $\lambda(G_i)$ dual to the $(n-1)$-simplex $G_i$, for every $i = h + 1, h + 2, \ldots, n + 1 - k$, $k_1 - k_2$.

Obviously, if $\lambda(F)$ denotes the edge of $\Lambda(K)$ dual to any $(n-1)$-simplex $F$ of $K$, then $T := (T' - \{\lambda(G_{n+1}), \ldots, \lambda(G_{n+1-\overline{k}_1-k_2})\}) \cup \{\lambda(F_1)\} \cup \{\lambda(F_{n+1}), \ldots, \lambda(F_{n+1-\overline{k}_1-k_2})\} \cup \{\lambda(F_{n+1}), \ldots, \lambda(F_{n+1-\overline{k}_1-k_2})\}$ is a maximal tree in $\Lambda(K)$.

Further, if we denote by $\Sigma$ (resp. $\Sigma'$) the boundary of the $n$-ball associated to $T$ (resp. $T'$) by step (b) of cone-algorithm applied to $K$ (resp. $L$), then it is easy to check that the
\( h - 1 \geq 0 \) pairs of twin \((n - 1)\)-simplices corresponding to the edges \( \lambda(F_2), \lambda(F_3), \ldots, \lambda(F_h) \) of \( \Lambda(K) \) constitute \( h - 1 \) (internal) \( 1 \)-dipoles in \( \Sigma \); moreover, in case \( k \geq 1 \), their elimination implies the pairs of boundary \((n - 1)\)-simplices \( \{ F_{nk+2}, F'_{nk+2} \}, \ldots, \{ F_{nk+1}, F'_{nk+1} \} \) to be \( k \) boundary dipoles in \( \Sigma \), whose elimination yields exactly \( \Sigma' \). Thus, the thesis directly follows from Lemma 5 (parts (a) and (b)): \( L' := A_\mathcal{I}_e(L) \) is obtained from \( K' := A_\mathcal{I}_e(K) \) by elimination of \( h - 1 \) internal dipoles and \( k \) boundary dipoles.

In order to deal with pseudocomplexes (and not only with simplicial complexes), a further fundamental result is needed: in one sense, it enables us to apply bistellar operation to the «simplicial part» of a pseudocomplex.

**Lemma 11.** Let \( K, K' \) be pseudosimplicial triangulations of the same \( n \)-manifold \( M^n \), with \( \partial K = \partial K' \). Then, a finite sequence of internal dipole moves and bistellar operations exists, yielding \( K' \) from \( K \).

**Proof.** First of all, note that bistellar \( n \)-operation is (obviously) well defined for any maximal simplex of a pseudocomplex; thus, we can consider the pseudosimplicial triangulation \( \tilde{K} \) (resp. \( \tilde{K}' \)) of \( M^n \) obtained from \( K \) (resp. \( K' \)) by applying \( n \)-bistellar operation to every \( n \)-simplex having a boundary \((n - 1)\)-face. It is easy to check that, if \( Q \) (resp. \( Q' \)) denotes the subcomplex of \( \tilde{K} \) (resp. \( \tilde{K}' \)) consisting of all \( n \)-simplices having a boundary \((n - 1)\)-face, with all their faces, then \( Q \) and \( Q' \) result to be isomorphic \( n \)-pseudocomplexes. Now, Proposition 6 (b) ensures the existence of a finite sequence of internal dipole insertions in \( \tilde{K} \) (resp. \( \tilde{K}' \)), giving rise to a pseudosimplicial triangulation \( \tilde{K}^∗ \) (resp. \( \tilde{K}'^* \)) of \( M^n \) with the property that every pair of internal simplices shares one only common face, if any: thus, \( \tilde{K}^∗ - Q \) and \( \tilde{K}'^* - Q' \) result to be simplicial triangulations of \( M^n \) with isomorphic boundaries (which are obtained from \( \partial K = \partial K' \) by applying an \((n - 1)\)-bistellar operation to every \((n - 1)\)-simplex). The thesis now directly follows from Proposition 2, since \( \tilde{K}^* - Q \) and \( \tilde{K}'^* - Q' \) (and hence \( \tilde{K}^* \) and \( \tilde{K}'^* \), too) are bistellar equivalent.

We are now able to prove that our Main Theorem holds for pseudosimplicial triangulations of the class \( \mathcal{K}_e \) having the same boundary; this will be the key-stone to prove it in the general situation.

**Proposition 12.** Let \( K, K' \in \mathcal{K}_e \), represent the same \( n \)-manifold \( M^n \). If \( \partial K = \partial K' \), then \( K \) and \( K' \) are equal up to internal dipoles.

**Proof.** It is easy to check that a (possibly void) finite sequence of internal \( 1 \)-dipole eliminations – all involving the «last» colour \( n \) – may be performed on the (colorable)
pseudocomplex $K \in \mathcal{K}_n$ (resp. $K' \in \mathcal{K}_n$), so that $K_1 \in \mathcal{C}(M^\sigma)$ (resp. $K'_1 \in \mathcal{C}(M')$) is obtained. Thus, Lemma 8 ensures the existence of $K_0 \in \mathcal{C}(M^\sigma)$ (resp. $K'_0 \in \mathcal{C}(M')$) and a suitable $L \in \mathcal{U}(K_0)$ (resp. $L' \in \mathcal{U}(K'_0)$) such that $K_1$ and $L$ (resp. $K'_1$ and $L'$) are equal up to internal (geometric) dipoles. Moreover, since $K_0$ and $K'_0$ are pseudosimplicial triangulations of the same $n$-manifold $M^\sigma$ coinciding on the boundary, they can be obtained from each other by a finite sequence of bistellar operations and internal dipole moves (Lemma 11); thus, applications of Lemma 9 and Lemma 10 yield $N \in \mathcal{U}(K_0)$ and $N' \in \mathcal{U}(K'_0)$, which are equal up to internal dipoles.

The thesis is now an easy consequence of Lemma 4: $L$ and $N$ (resp. $L'$ and $N'$) are equal up to internal dipoles, and hence $K$ and $K'$ are, too. 

Since both cone algorithms and bistellar operations do not affect the boundary triangulations, in the general situation it is useful to know how to induce boundary moves on a manifold triangulation. The problem is solved by means of the following result, which can be seen as a «translation» in terms of pseudosimplicial triangulations of an analogue statement concerning coloured graphs (see [C2; Lemma 9]).

**Lemma 13.** Let $H, H'$ be pseudosimplicial triangulations of a closed $(n-1)$-manifold. If $H$ and $H'$ are equal up to (geometric) dipoles, then for every pseudosimplicial triangulation $K$ of an $n$-manifold $M^\sigma$, with $\partial K = H$, there exists a pseudosimplicial triangulation $K'$ of $M^\sigma$, with $\partial K' = H'$ and such that $K$ and $K'$ are equal up to (geometric) dipoles.

**Proof.** Obviously, it is sufficient to prove the statement in the following two cases:

- case (a): $H'$ is obtained from $H$ by a (geometric) dipole elimination;
- case (b): $H'$ is obtained from $H$ by a (geometric) dipole insertion.

Case (a): Let $H'$ be obtained from $H$ by eliminating the (geometric) $h$-dipole $\mathcal{D} = \{ \bar{\sigma}, \bar{\tau} \}$ ($1 \leq h \leq n - 1$); if $\sigma$ (resp. $\tau$) are the (boundary) $n$-simplex of $K$ containing $\bar{\sigma}$ (resp. $\bar{\tau}$), then the required pseudosimplicial triangulation $K'$ of $M^\sigma$ is simply obtained from $K$ by identifying the boundary faces $\bar{\sigma}$ and $\bar{\tau}$ of $\sigma$ and $\tau$.

For, let $\overline{K}$ be the pseudocomplex obtained from $K$ by considering the (colourable) pseudocomplex $w^* (\mathcal{D} \cup \mathcal{D}')$, where $\mathcal{D}' = \{ \bar{\sigma}', \bar{\tau}' \}$ and $\mathcal{D}'' = \{ \bar{\sigma}'', \bar{\tau}'' \}$ are two disjoint copies of $\mathcal{D} = \{ \bar{\sigma}, \bar{\tau} \}$ and $w$ is a new vertex, by identifying the face $\bar{\sigma}'$ (resp. $\bar{\tau}'$) of $w^* \sigma'$ (resp. $w^* \tau'$) with the face $\bar{\sigma}$ (resp. $\bar{\tau}$) of $\sigma$ (resp. $\tau$), and by identifying each one of the other $n-h$ faces of $w^* \sigma'$ (resp. $w^* \tau'$) with the corresponding one of $w^* \sigma''$ (resp. $w^* \tau''$).

It is now easy to check that $K$ (resp. $K'$) may be obtained from $\overline{K}$ by eliminating the dipole $\{ \sigma', \sigma'' \}$ of type $n-h$ (resp. $\{ \sigma'', \tau'' \}$ of type $h$) and the resulting dipole $\{ \tau', \tau'' \}$ of type $n$ (resp. $\{ \sigma', \tau' \}$ of type $n$). Since $\partial K' = H'$ is obviously verified, in case (a) the thesis follows.

Case (b): Let $H'$ be obtained from $H$ by adding the $h$-dipole $\mathcal{D} = \{ \bar{\sigma}, \bar{\tau} \}$ within the
disjoint star $Z$ of an $(h-1)$-simplex of $H$ (i.e.: within $n-h$ suitable $(n-2)$-simplices of $H$ having a common $(h-1)$-simplex of $H$).

Further, let $\overline{K}$ be the pseudocomplex obtained from $K$ by considering the (colourable) pseudocomplex $w^*(Z' \cup Z'')$, where $Z'$ and $Z''$ are two disjoint copies of $Z$ and $w$ is a new vertex, by identifying each face of $Z'$ in $w^*Z'$ with the corresponding one of $Z$ in $\partial K$, and by identifying each one of the other faces of $w^*Z'$ with the corresponding one of $w^*Z''$.

Since, for every $(n-1)$-simplex $\overline{\partial}Z$ of $Z$, $\{w^*\rho', w^*\rho''\}$ results to be an $(n-h)$-dipole in $\overline{K}$, it is easy to check that $K$ may be obtained from $\overline{K}$ by $p$ dipole eliminations (of type $m$, with $1 \leq m \leq n$), where $p$ is the number of $(n-1)$-simplices of $Z$.

On the other hand, $\partial \overline{K} = \partial K = H$, and a dipole $\overline{D}$ isomorphic with $D$ may be added to $\overline{K}$ within the subcomplex $w^*Z''$ (which is the disjoint star of a boundary $(h-1)$-simplex of $\overline{K}$), giving rise to a pseudosimplicial triangulation $K'$ of $M^r$ with the required properties: $\partial K' = H'$ and $K, K'$ equal up to dipoles. □

Lemma 14. Let $K, K'$ be pseudosimplicial triangulations of the same $n$-manifold $M^r$, with $\partial K, \partial K' \in \mathscr{K}_{n-1}$. Then, a finite sequence of dipole moves and bistellar operations exists, yielding $K'$ from $K$.

Proof. Since $\partial K, \partial K' \in \mathscr{K}_{n-1}$, Proposition 12 ensures the existence of a finite sequence of (internal) dipoles, yielding $\partial K'$ from $\partial K$. Then, Lemma 13 yields a pseudosimplicial triangulation $K''$ of $M^r$, with $\partial K'' = \partial K$ and such that $K'$ and $K''$ are equal up to dipoles. At this point, Lemma 11 ensures $K$ and $K''$ to be equal up to internal dipoles and bistellar operations; thus, the thesis directly follows. □

Proof of the Main Theorem. Since it is known that dipole moves do not affect the homeomorphism class of the represented manifold (recall Proposition 1), one only implication has to be proved.

For, let us assume $K$ and $K'$ to be (different) pseudosimplicial triangulations of the same manifold $M^r$. If $\partial K = \partial K'$ (for example, if $K$ and $K'$ are both pseudocomplexes with void boundary), then Proposition 12 yields the thesis. Otherwise, the proof follows the same arguments as the one of Proposition 12, by making use of Lemma 14 instead of Lemma 11. We include here the entire proof, for sake of completeness.

It is easy to check that a (possibly void) finite sequence of 1-dipole eliminations – all involving the «last» colour $n$ – may be performed on the (colorable) pseudocomplex $K \in \mathscr{K}_n$ (resp. $K' \in \mathscr{K}_n$), so that $K_i \in \mathcal{C}(M^r)$ (resp. $K'_i \in \mathcal{C}(M^r)$) is obtained. Thus, Lemma 8 ensures the existence of $K_0 \in \mathcal{C}(M^r)$ (resp. $K'_0 \in \mathcal{C}(M^r)$) and a suitable $L \in \mathfrak{l}(K_i)$ (resp. $L' \in \mathfrak{l}(K'_i)$) such that $K_i$ and $L$ (resp. $K'_i$ and $L'$) are equal up to internal (geometric) dipoles. At this point, we note that $K_0$ and $K'_0$ are pseudosimplicial triangulations of the same $n$-manifold $M^r$, with $\partial K_0 = \partial K'_0 \in \mathscr{K}_{n-1}$ and $\partial K_i = \partial K'_i$; hence, Lemma 14 ensures that they can be obtained from each other by a finite sequence of bistellar operations;
operations and internal dipole moves. Then, applications of Lemma 9 and Lemma 10 yield $N \in U(K_0)$ and $N' \in U(K'_0)$, which are equal up to internal dipoles. The thesis is now an easy consequence of Lemma 4: $L$ and $N$ (resp. $L'$ and $N'$) are equal up to dipoles, and hence $K$ and $K'$ are, too.

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